

UNIVERSITY OF LUND

LUND INSTITUTE OF TECHNOLOGY

1977:1

LTH

ON BESOV SPACES

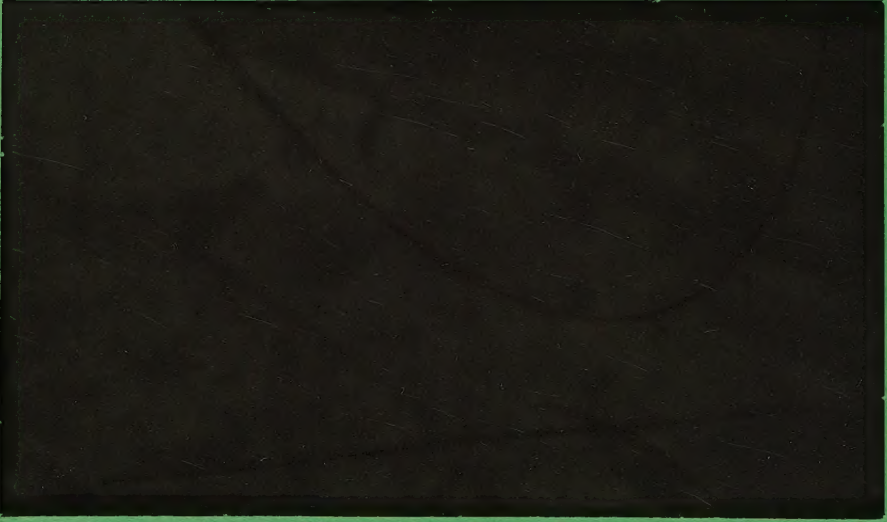
by Björn Jawerth



DEPARTMENT OF MATHEMATICS

S-220 07 Lund 7, Sweden

1977



1977:1

LTH

ON BESOV SPACES

by Björn Jawerth



By due permission of the Faculty of Technology
of the University of Lund, to be publicly discussed
in Matematik Huset, Sal C, on February 4, 1977, at
10 a.m., for the examination of Doctor of Technology.

1911

1911

ON THE

BY



By the permission of the Faculty of Technology
of the University of Leiden, to be publicly discussed
in the Institute of Technology, 2nd floor, on February 1, 1911, at
10 a.m., for the examination of the Faculty of Technology.

The subjects of this dissertation were suggested by Professor Jaak Peetre to whom the author is deeply grateful for his generously given advice and never failing interest.

Lund, January 1977

Björn Jawerth

Papers summarized in this dissertation

- [A] Weighted norm inequalities for functions of exponential type. To appear in Ark. Mat.
- [B] Some observations on Besov and Lizorkin-Triebel spaces. To appear in Math. Scand.
- [C] The trace of Sobolev and Besov spaces if $0 < p < 1$. To appear in Stud. Math.

On Besov spaces

We will deal with a number of questions related to Besov spaces $\dot{B}_p^{sq}$, with special emphasis on the case $0 < p < 1$. We begin by a sketch of the historical origin of the problems.

Sobolev and Besov spaces appear in many branches of analysis, for instance in partial differential equations, approximation theory, Fourier analysis and numerical analysis.

Sobolev spaces W_p^k were formally introduced by Sobolev and extensively studied by him in the late 30's (see [16]), but their history can be traced much further back (beginning with Riemann and Dirichlet's principle).

Besov spaces B_p^{sq} appear two decades later after pioneering work by Nikolskij (see [11]). A comprehensive study of their properties was made in 1959 by Besov [2] using finite differences. In the one-dimensional case the germ of this theory can however be found in the work of Hardy and Littlewood (c:a 1930) on functions on the unit circle, centering around the study of the Hardy classes H_p . The original approach by Hardy-Littlewood was extended to the case of $\mathbb{R}^n$ by Taibleson [19]. The theory of Hardy classes on the other hand was extended to the case of $\mathbb{R}^n$ by Stein-Weiss [18], using M. Riesz' generalization of the notion of conjugate function. A third method of definition of Besov spaces, having certain advantages, has been advocated by Peetre (see [12]; cf. also Löffström [9] and for $p=2, q=2$ Hörmander [7]). Since it is the one used here, we briefly recall it in the case of so called "homogeneous" spaces:

Choose a sequence of testfunctions $\{\varphi_\nu\}_{\nu \in \mathbb{Z}}$ with $\varphi_\nu(x) = 2^{\nu n} \varphi(2^\nu x)$ where $\varphi \in \mathcal{S}$, $\text{supp } \varphi = \{2^{-1} \leq |\xi| \leq 2\}$ and $\sum \varphi_\nu = \delta$. Then we set

$$\dot{B}_p^{sq} = \{f \in \mathcal{S}' : (\sum_\nu (2^{\nu s} \|\varphi_\nu * f\|_{L_p})^q)^{1/q} < \infty\}.$$

("Homogeneous" Besov spaces)

"Inhomogeneous" Besov spaces B_p^{sq} are defined in a similar way by chopping off the sum at $\nu=0$. It is for technical reasons we prefer

to work mostly with "homogeneous" spaces.

Besides the spaces $\dot{B}_p^{sq}$ we shall also be concerned with certain other spaces $\dot{F}_p^{sq}$, introduced only comparatively recently by Lizorkin [8] and studied by Triebel [20]. The definition reads

$$\dot{F}_p^{sq} = \{f \in \mathcal{S}' : \|(\sum_{\nu} |2^{\nu s} \varphi_{\nu} * f|^q)^{1/q}\|_{L_p} < \infty\}.$$

(Lizorkin-Triebel spaces)

It is easy to see that

$$\dot{B}_p^{sp} = \dot{F}_p^{sp}.$$

On the other hand, using Littlewood-Paley theory one can prove that

$$\dot{F}_p^{02} = H_p$$

where H_p is the Stein-Weiss generalization of the Hardy classes.

In the above discussion we have tacitly ignored the question of the range of the parameters. Classically it was always assumed that $s > 0$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$. The extension to arbitrary s appears rather early, in which the elements of our spaces no longer are functions but distributions. $0 < q \leq \infty$ is also rather straightforward (see e.g. [13]). The extension to $0 < p \leq \infty$ was undertaken by Flett [6] using the Hardy-Littlewood-Taibleson approach and by Peetre [14] using the present definition. This is by no means trivial since L_p , $0 < p < 1$, as a topological vector space behaves very badly. For instance, its dual is 0 so it cannot be imbedded in any space of distributions. The reason why a nice theory of Besov spaces with $0 < p < 1$ still can be developed is that, with the present definition, the base space is really not L_p but H_p .

We shall answer several questions for $0 < p < 1$ left open by Peetre. In this context the well-known work by Fefferman-Stein [5], opening up a quite new vista on H_p spaces, plays an important role. Not only do these authors manage to determine the dual of H_1 but - and this is what is important to us - they obtain a purely real variable characterization of H_p .

In Peetre's approach to Besov spaces with $0 < p < 1$ a certain theorem by Plancherel-Polya [15] on functions of exponential type plays a

crucial role. In [1] it was noted that an alternative proof of this theorem could be based on the ideas of Fefferman-Stein. In [A] we therefore try to extend the argument to the case when the Lebesgue measure on $\mathbb{R}^n$ is replaced by a general positive measure μ : Let $\mathbb{E}_r$ be the functions on $\mathbb{R}^n$ of exponential type $\leq r$. Given $f \in \mathbb{E}_r \cap L_p(d\mu)$ we ask when it is true that the maximum of f over balls can be estimated by f itself. It turns out that the relevant condition on μ is

$$\mu(2Q) \leq C \mu(Q)$$

for all cubes $Q \subset \mathbb{R}^n$ ($2Q$ being the cube with the same center but double side). This condition should be compared with the one used by Muckenhoupt [10] (cf. also [3]).

Originally we planned to apply this result to a study of Besov spaces with weights, but we have at least temporarily abandoned this plan.

In [B] we prove several imbedding theorems for the spaces $\dot{B}_p^{sq}$ and $\dot{F}_p^{sq}$. One application is a characterization of the dual spaces for $0 < p < 1$. We find

$$(\dot{B}_p^{sq})' \approx \begin{cases} \dot{B}_\infty^{-s+n(1/p-1), \infty} & \text{if } 0 < q \leq 1 \\ \dot{B}_\infty^{-s+n(1/p-1), q} & \text{if } 1 < q \end{cases}$$

Here the case $0 < q \leq 1$ is due to Flett and Peetre but $1 < q < \infty$ is new, as well as

$$(\dot{F}_p^{sq})' \approx \dot{B}_\infty^{-s+n(1/p-1), \infty} \quad \text{if } 0 < q < \infty.$$

This includes

$$(H_p)' \approx \dot{B}_\infty^{n(1/p-1), \infty}$$

which is the analogue of the famous result by Duren-Romberg-Shields for the circle (see [4]; cf. [5], [21], [14]). Another application is the following extension of Hardy's inequality (for H_p on the circle see [4]):

$$\left(\int |\hat{f}(\xi)|^p / |\xi|^{n(2-p)} d\xi \right)^{1/p} \leq C \|f\|_{\dot{F}_p^{0q}}.$$

In [B] and [C] we consider the problem of the trace on $\mathbb{R}^{n-1}$ of our spaces of distributions on $\mathbb{R}^n$. Classically it is known that the trace of $\dot{B}_p^{sq}(\mathbb{R}^n)$, where $1 \leq p \leq \infty$, exists and is $\dot{B}_p^{s-1/p, q}(\mathbb{R}^{n-1})$ provided $s > 1/p$. The precisely analogous result is now established for $0 < p < 1$, the only significant difference being that $s > 1/p$ must now be replaced by $s > 1/p + (n-1)(1/p-1)$ (which is the optimal result). We also prove that the trace of $\dot{F}_p^{sq}(\mathbb{R}^n)$ exists and is $\dot{B}_p^{s-1/p, p}(\mathbb{R}^{n-1})$ under the same restriction on s . This includes in particular theorems for potential spaces. (For references see Stein [17] ch. 6.)

Finally, we mention that we have proved that $\dot{B}_p^{sq}$ as a topological vector space is isomorphic to $\ell_q(\ell_p)$. We use the Plancherel-Polya theorem but since otherwise the proof is easily deduced from the arguments of Peetre in [12], we have not bothered to write down the details.

REFERENCES

- [1] J. Bergh - J. Peetre, On the spaces V_p ($0 < p \leq \infty$).
Boll. Un. Mat. Ital., 3 (1970), pp. 632-648.
- [2] O.V. Besov, On a certain family of functional spaces.
Imbedding and extension theorems. Dokl. Akad. Nauk
SSSR, 126 (1959), pp. 1163-1165 (Russian).
- [3] R. Coifman - C. Fefferman, Weighted norm inequalities
for maximal functions and singular integrals. Studia
Math., 51 (1974), pp. 241-250.
- [4] P.L. Duren, Theory of H^p spaces. Academic Press, New
York - London, 1970.
- [5] C. Fefferman - E.M. Stein, H^p spaces of several variables.
Acta Math., 129 (1972), pp. 137-193.
- [6] T.M. Flett, Lipschitz spaces of functions on the circle
and the disc. J. Math. Anal. Appl., 39 (1972), pp.
125-158.
- [7] L. Hörmander, Linear partial differential operators.
Springer-Verlag, Berlin - Heidelberg - New York, 1969.
- [8] P.I. Lizorkin, Operators, connected with fractional
differentiation, and classes of differentiable functions.
Trudy Mat. Inst. Steklov, 117 (1972), pp. 212-243
(Russian).
- [9] J. Löfström, Besov spaces in theory approximation.
Ann. Math. Pura Appl., 85 (1970), pp. 93-184.
- [10] B. Muckenhoupt, Weighted norm inequalities for classical
operators. (To appear.)
- [11] S.M. Nikolskij, Approximation of functions of several
variables and embedding theorems. Springer-Verlag,
Berlin - Heidelberg - New York, 1974.
- [12] J. Peetre, New thoughts on Besov spaces. Duke University
Press, Durham, to appear.

- [13] J. Peetre, Remark on the dual of an interpolation space. Math. Scand., 34 (1974), pp. 124-128.
- [14] J. Peetre, Remarques sur les espaces de Besov. Le cas $0 < p < 1$. C. R. Acad. Sci. Paris, 277 (1973), pp. 947-949.
- [15] M. Plancherel - G. Polya, Fonctions entières et intégrales de Fourier multiples. Comment. Math. Helv., 9 (1937), pp. 224-248.
- [16] S.L. Sobolev, Some applications of functional analysis in mathematical physics. Leningrad, 1950 (Russian).
- [17] E.M. Stein, Singular integrals and differentiability properties of functions. Princeton University Press, Princeton, 1970.
- [18] E.M. Stein - G. Weiss, On the theory of harmonic functions of several variables I: the theory of H^p spaces. Acta Math., 103 (1960), pp. 25-62.
- [19] M.H. Taibleson, On the theory of Lipschitz spaces of distributions on Euclidean n -space, I-III. J. Math. Mech., 13 (1964), pp. 407-480; 14 (1965), pp. 821-840; 15 (1966), pp. 973-981.
- [20] H. Triebel, Spaces of distributions of Besov type on Euclidean n -space. Duality, interpolation. Ark. Mat., 11 (1973), pp. 13-64.
- [21] T. Walsh, The dual of $H^p(\mathbb{R}_+^{n+1})$ for $p < 1$. Can. J. Math., 25 (1973), pp. 567-577.

